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Stability of a Homopolar device

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In this paper we have studied the problem of instability in a Homopolar device. We consider a plasma shell of finite thickness and of infinite conductivity to be in steady state in the presence of a space varying magnetic field such as found in Ixion device. The instabilities for such a device have been studied by normal mode technique both for axisymmetric and azimuthal disturbances. It has been found that the system is unstable in zeroth order of the perturbation parameter β for an axisymmetric disturbance. Further, the effect of slow rotation and slowly varying magnetic field is to increase the growth rate of instability. The system is unstable in the zeroth order for the azimuthal disturbance. The first order dispersion relation shows that the effects of low intensity ring currents is to increase the growth rate of instability. Further, the increase of rotation decreases the growth rate of instability.

Introduction

The study of instabilities encountered in plasma systems is of considerable importance in connection with fusion devices and in astrophysical context. A number of investigations has been made by several authors under varying assumptions to get a magnetic trap which confines a plasma for sufficiently long time. The study of instabilities of such devices is also very important in order to find if these devices are useful for controlled fusion.

In the present paper we have studied the instabilities which set in an idealized laboratory device known as Ixion or Homopolar device. In this a plasma shell of very small resistivity is immersed in radial electric and axial magnetic fields which decrease as the radial coordinate increases. The effect of crossed fields is to give each particle a drift perpendicular to both electric and magnetic fields. An experimental and theoretical speculation of this device for containment has been made by Boyer *et al.* (1958). They have shown that particle containment is improved by the centrifugal force of crossed field rotation which has the tendency to keep the particles away from the axis. Further, Anderson *et al* (1958) have also investigated the characteristics of this experimental device. Perkins

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& Post (1963) while investigating the MHD stability of a cylindrical plasma have found that the system is unstable whenever rotation is present. Verma & Verma (1965) have also invesitgated the stability of a rotating plasma cylinder and have shown that rotation has a stabilizing influence

In the present investigation we have applied the usual normal mode technique to study the stability of the device In section 2 we have recorded the steady state In the subsequent sections we have recorded the solutions for axisymmetric and azimuthal disturbance and the corresponding dispersion relations obtained after using the required boundary conditions which are discussed in detail in the last two sections.

2 Steady state

The non-dimensional magnetic field in the steady state is given by

$$\boldsymbol{B}^{(i)} = [0, 0, (1-\beta/r^2)^{1/4}], \quad 1 \leqslant r \leqslant m$$

$$\boldsymbol{B}^{(o)} = [0, 0, H_1], \quad 0 \leqslant r \leqslant 1$$

and

$$\boldsymbol{B}^{(o)} = [0, 0, H_2], \quad r \geqslant m$$

where the superscripts i and o stand for inside and outside of the plasma device, respectively. The electric field inside the plasma region is

$$\boldsymbol{E}^{(i)} = [K/r, 0, 0] \quad 1 \leqslant r \leqslant m$$

and outside the plasma region it is zero. K is the rotation parameter and m is the ratio of the outer to inner radius The corresponding pressure and velocity can be calculated through the momentum equation and generalized Ohm's law It is found that due to the presence of electric field in the radial direction inside the plasma region the azimuthal component of the velocity is non-zero

3 Axisymmetric disturbances

We perturb the steady state described in section 2 in an axisymmetric manner and assume that these disturbances are so small that their squares and products can be neglected In order to investigate the stability of such a system we assume that these perturbed quantities vary with time and z-coordinate exponentially *i.e.*, as $e^{i\omega t + ilz}$ where ω is the angular frequency and l is the axial wave number We denote the amplitudes of the perturbations in velocity, pressure, magnetic field and the electric field, respectively, by $\boldsymbol{v}$, $\tilde{p}$, $\boldsymbol{b}$ and $\boldsymbol{e}$. We consider β to be small and treat it as an expansion parameter so that

$$X = X_0 + \beta X_1 + \dots \tag{3.1}$$

where X is taken as $\boldsymbol{v}$, $\tilde{p}$, $\boldsymbol{b}$, $\boldsymbol{e}$ and ω. In view of (3.1) and using the governing momentum and Maxwell's equations, we obtain the following equation for v_{r0} in the zeroth order after linearization

$$\frac{d^2v_{r0}}{dr^2}+\frac{1}{r}\frac{dv_{r0}}{dr}-(l^2+1/r^2)\,v_{r0}=0 \tag{3.2}$$

This equation has been obtained on the approximation of slow rotation such that K can be treated to be of the same order as β. After solving (3.2) for v_{r0} and making use of the resulting solution in the remaining set of equations we find the following zeroth order solutions

$$\begin{aligned}
v_{r0} &= AI_1(lr)+BK_1(lr),\\
v_{\theta 0} &= 0,\\
v_{z0} &= i[AI_0(lr)-BK_0(lr),]\\
b_{r0} &= \frac{l}{\omega_0}[AI_1(lr)+BK_1(lr)],\\
b_{\theta 0} &= 0,\\
b_{z0} &= \frac{il}{\omega_0}[AI_0(lr)-BK_0(lr)],\\
p_0 &= -\frac{i\omega_0}{l}[AI_0(lr)-BK_0(lr)]\\
e_{r0} &= 0,\\
e_{\theta 0} &= AI_1(lr)+BK_1(lr),\\
e_{z0} &= 0.
\end{aligned} \tag{3.3}$$

where A and B are arbitrary constants of integration. The governing differential equation for v_{r1} is given by

$$\frac{d^2v_{r1}}{dr^2}+\frac{1}{r}\frac{dv_{r1}}{dr}-\left(l^2+\frac{1}{r^2}\right)v_{r1}=\frac{l^3}{r^3(\omega_0^2-l^2)}[AI_0(lr)-BK_0(lr)], \tag{3.4}$$

Solving (3.4) and taking care of the set of equations involving other physical quantities, we obtain the following set of first order solutions:

$$\begin{aligned}
v_{r1} &= A_{11}I_1(lr)+B_{11}K_1(lr)+K_1(lr)I_1-I_1(lr)I_2,\\
v_{\theta 1} &= 0,\\
v_{z1} &= i[A_{11}I_0(lr)-B_{11}K_0(lr)-K_0(lr)I_1-I_0(lr)I_2],
\end{aligned}$$

$$b_{r1} = \frac{1}{\omega_0}\Big[lA_{11}I_1(lr)+lB_{11}K_1(lr)+lK_1(lr)I_1-lI_1(lr)I_2$$

$$-\frac{l}{4r^2}\{AI_1(lr)+BK_1(lr)\}-\frac{\omega_1 l}{\omega_0}\{AI_1(lr)+BK_1(lr)\}\Big],$$

$$b_{\theta 1} = 0,$$

$$b_{z1} = \frac{il}{\omega_0}\Big[A_{11}I_0(lr)-B_{11}K_0(lr)-lI_0(lr)I_2-lK_0(lr)I_1$$

$$+\frac{1}{2r^3 l}\{AI_0(lr)+BK_1(lr)\}-\left(\frac{1}{4r^2}+\frac{\omega_1}{\omega_0}\right)\{AI_0(lr)-BK_0(lr)\}\Big],$$

$$p_1 = \frac{1}{il}\Big[\omega_0\{A_{11}I_0(lr)-B_{11}K_0(lr)-lK_0(lr)I_1-lI_0(lr)I_2\}$$

$$+\omega_1\{AI_0(lr)-BK_0(lr)\}+\frac{l}{2r^3\omega_0}\{AI_1(lr)+BK_1(lr)\}\Big],$$

$$e_{r1} = -\frac{b_{z0}}{r},$$

$$e_{\theta 1} = A_{11}I_1(lr)+B_{11}K_1(lr)+lK_1(lr)I_1-lI_1(lr)I_2-\frac{1}{4r^2}[AI_1(lr)+BK_1(lr)],$$

and

$$e_{z1} = -\frac{b_{r0}}{r} \tag{3.5}$$

where A_{11}, B_{11} are arbitrary constants and the integrals I_1 and I_2 are given by

$$I_1 = \int^{lr} I_1(x)\frac{l^4}{x^2(l^2-\omega_0^2)}\{AI_0(x)-BK_0(x)\}dx$$

and

$$I_2 = \int^{lr} K_1(x)\frac{l^4}{x^2(l^2-\omega_0^2)}\{AI_0(x)-BK_0(x)\}dx.$$

4. Solutions in vacuum

We have to solve Maxwell's field equations in the outside and inside vacua. The set of zeroth order solutions is

$$\left.\begin{array}{lll} b_{r0} = lCI_1(lr), & b_{\theta 0} = 0, & b_{z0} = ilCI_0(lr), \\ e_{r0} = -iEI_1(lr), & e_{\theta 0} = \omega_0 CI_1(lr), & e_{z0} = EI_0(lr) \end{array}\right\} 0 \leqslant r \leqslant 1$$

and

$$\left.\begin{array}{lll} b_{r0} = -lDK_1(lr), & b_{\theta 0} = 0, & b_{z0} = ilDK_0(lr), \\ e_{r0} = iFK_1(lr), & e_{\theta 0} = -\omega_0 DK_1(lr), & e_{z0} = FK_0(lr) \end{array}\right\} r \geqslant m \tag{4.1}$$

where C, D, E and F are arbitrary constants of integration. The set of first order solutions is

$$\left.\begin{aligned} b_{r1} &= C_{11}lI_1(lr), & e_{r1} &= -iE_{11}I_1(lr) \\ b_{\theta 1} &= 0\,, & e_{\theta 1} &= \omega_0 C_{11}I_1(lr)+\omega_1 CI_1(lr) \\ b_{z1} &= ilC_{11}I_0(lr), & e_{z1} &= E_{11}I_0(lr) \end{aligned}\right\} 0 \leqslant r \leqslant 1$$

and

$$\left.\begin{aligned} b_{r1} &= -D_{11}lK_1(lr), & e_{r1} &= iF_{11}K_1(lr) \\ b_{\theta 1} &= 0\,, & e_{\theta 1} &= -\,\omega_0 D_{11}K_1(lr)-\omega_1 DK_1(lr) \\ b_{z1} &= ilD_{11}K_0(lr), & e_{z1} &= F_{11}K_0(lr) \end{aligned}\right\} r \geqslant m \qquad (4.2)$$

where C_{11}, D_{11}, E_{11} and F_{11} are arbitrary constants of integration.

5. Two-dimensional disturbances

Starting with the steady state described in section 2 we have applied small amplitude disturbances of the type $\exp(i\omega t+in\theta)$ to the system where n is the wave number and ω is the angular frequency After adopting the same procedure as done earlier but keeping K finite we obtain the following set of solutions :

(*i*) *Zeroth order solutions*

$$v_{r0} = \left(Ar^{n-1}+\frac{B}{r^{n+1}} \right),$$

$$v_{\theta 0} = i\left(Ar^{n-1}-\frac{B}{r^{n+1}} \right),$$

$$v_{z0} = 0,$$

$$b_{r0} = b_{\theta 0} = b_{z0} = 0$$

$$p_0 = i\left(\frac{K}{r^2}-\frac{\omega_0}{n} \right)\left(Ar^n-\frac{B}{r^n} \right),$$

$$e_{r0} = i\left(-Ar^{n-1}+\frac{B}{r^{n+1}} \right),$$

$$e_{\theta 0} = v_{r0},$$

$$e_{z0} = 0. \qquad (5.1)$$

(ii) *First order solutions*

$$v_{r1} = Cr^{n-1} + \frac{D}{r^{n+1}} - r^{-n-1}I_3 + r^{n-1}I_4,$$

$$v_{\theta 1} = i\left(Cr^{n-1} - \frac{D}{r^{n+1}} + r^{-n-1}I_3 + r^{n-1}I_4 \right),$$

$$v_{z1} = 0,$$

$$b_{r1} = b_{\theta 1} = 0,$$

$$b_{z1} = \frac{i}{2}\left(\frac{Ar^{n-2}+Br^{-n-2}}{\omega_0 r^2 - nK} \right.$$

$$p_1 = \left(\frac{iK}{r} - \frac{i\omega_0 r}{n} \right)\left(Cr^{n-1} - \frac{D}{r^{n+1}} + r^{-n-1}I_3 + r^{n-1}I_4 \right)$$

$$+ \left(\frac{iK}{4r^3} - \frac{i\omega_1 r}{n} \right)\left(Ar^{n-1} - \frac{B}{r^{n+1}} \right)$$

$$+ \frac{iK}{2n}\left(Ar^{n-1} + \frac{B}{r^{n+4}} \right) + \frac{ir}{2K} R(r).$$

$$e_{r1} = -i\left[\frac{R(r)}{2} - \frac{Ar^{n-3}}{4} + \frac{B}{4r^{n+3}} + Cr^{n-1} - \frac{D}{r^{n+1}} + r^{-n-1}I_3 + r^{n-1}I_4 \right],$$

$$e_{\theta 1} = -\frac{1}{4r^2}\left(Ar^{n-1}+Br^{-n-1}\right) + \left(Cr^{n-1}+Dr^{-n-1}-r^{-n-1}I_3+r^{n-1}I_4\right),$$

$$e_{z1} = 0 \tag{5.2}$$

where,

$$I_3 = \int^r x^n R(x)dx,$$

$$I_4 = \int^r x^{-n} R(x)dx$$

and

$$R(r) = \frac{Ar^{n-3}+Br^{-n-3}}{\left(n - \frac{\omega_0 r^2}{K} \right)}.$$

6. Solutions in vacuum

Solving Maxwell's field equations in the outside vacuum we obtain the following sets of zeroth and first order solutions

(i) *Zeroth order solutions*

$$\left.\begin{aligned} b_{r0} &= -iE_0 r^{n-1}, & b_{\theta 0} &= E_0 r^{n-1}, & b_{z0} &= 0, \\ e_{r0} &= -iM_0 r^{n-1}, & e_{\theta 0} &= M_0 r^{n-1}, & e_{z0} &= \frac{iE_0\omega_0 r^n}{n} \end{aligned}\right\} 0 \leqslant r \leqslant 1$$

and

$$\left.\begin{aligned} b_{r0} &= iF_0 r^{-n-1}, & b_{\theta 0} &= F_0 r^{-n-1}, & b_{z0} &= 0 \\ e_{r0} &= iN_0 r^{-n-1}, & e_{\theta 0} &= N_0 r^{-n-1}, & e_{z0} &= -\frac{i\omega_0 F_0 r^{-n}}{n} \end{aligned}\right\} r \geqslant m \tag{6.1}$$

(ii) *First order solutions*

$$\left.\begin{aligned} b_{r1} &= -iE_1 r^{n-1}, & b_{\theta 1} &= E_1 r^{n-1}, & b_{z1} &= 0 \\ e_{r1} &= -iM_1 r^{n-1}, & e_{\theta 1} &= M_1 r^{n-1}, & e_{z1} &= \frac{ir^n}{n}(\omega_1 E_0 + \omega_0 E_1) \end{aligned}\right\} 0 \leqslant r \leqslant 1$$

and

$$\left.\begin{aligned} b_{r1} &= iF_1 r^{-n-1}, & b_{\theta 1} &= F_1 r^{-n-1}, & b_{z1} &= 0, \\ e_{r1} &= iN_1 r^{-n-1}, & e_{\theta 1} &= N_1 r^{-n-1}, & e_{z1} &= -\frac{ir^{-n}}{n}(\omega_0 F_1 + \omega_1 F_0) \end{aligned}\right\} r \geqslant m \tag{6.2}$$

7. Dispersion relations for axisymmetric disturbance and discussion on them

After applying the boundary conditions at the the perturbed surfaces $r = 1 + (\delta r)\exp(i\omega t + ilz)$ and $r = m + (\delta r)\exp(i\omega t + ilz)$ we obtain the following zeroth and first order dispersion relations, respectively

$$\begin{aligned} &\omega_0^4[K_0(ml)K_1(ml)I_0(l)I_1(l) - I_0(ml)K_1(ml)I_1(l)K_0(l)] \\ &+\omega_0^2[K_0(ml)K_1(ml)\{-l^2I_0(l)I_1(l) + l^2H_1^2I_0(l)I_1(l)\} \\ &+l^2I_0(l)I_1(l)\{H_2^2K_0(ml)K_1(ml) - K_0(ml)K_1(ml)\} \\ &+l^2I_0(ml)K_1(ml)\{K_0(l)I_1(l) + H_1^2I_0(l)K_1(l)\}] \\ &+[\{l^2H_2^2K_0(ml)K_1(ml) - l^2K_0(ml)K_1(ml)\} \\ &\times\{l^2H_1^2I_0(l)I_1(l) - l^2I_0(l)I_1(l)\} \\ &-\{l^2K_0(l)I_1(l) + l^2H_1^2I_0(l)K_1(l)\} \\ &\times\{l^2I_0(ml)K_1(ml) + l^2H_2^2I_1(ml)K_0(ml)\}] = 0 \end{aligned}$$

i.e.,
$$a\omega_0^4 + b\omega_0^2 + c = 0 \tag{7.1}$$

and

$$\omega_1 = f(\omega_0, l, H_1, H_2, m). \tag{7.2}$$

We shall calculate ω_0 from (7.1) for given values of l, m, H_1 and H_2 and the corresponding ω_1 from (7 2) which gives the effect of small rotation and the slow space variation of magnetic field. We have discussed the following cases :

(*a*) *Large wave-number disturbances* :—In this case (7.1) reduces to

$$(\omega_0')^4-(1+H_1^2)(\omega_0')^2+(H_1^2+1)(H_2^2+1) = 0, \tag{7.3}$$

where $\omega_0 = \omega_0' l$ and $H_2^2 = H_1^2+K^2(1-1/m^2)$ From (7.3) we find that the system is always unstable. Further the effect of slow rotation and slowly varying field is to increase the growth rate of instability.

(*b*) *Small wave number disturbances* :—In this case (7.1) gives two growing modes and two decaying modes. Therefore the system is unstable From (7.2) we find that the inhomogeneity in the magnetic field enhances the growth rate of instability by a multiple of ω_1

Thus a Homopolar device is unstable against axisymmetric disturbances.

8. Dispersion relations for azimuthal disturbance and discussion on them

After using the boundary conditions we obtain the following dispersion relations of zeroth and first order respectively :

$$\begin{aligned}
&\omega_0^4\{m^{2n+4}-m^4\}+\omega_0^3\{-nK(3+H_1)m^{2n+4}-nK(3+H_2)m^{2n+2}\\
&+nK(3-H_2)m^2+nK(3-H_1)m^4\}+\omega_0^2\{(3n^2K^2-2nK^2)m^{2n+4}\\
&\quad m^{2n+2}(3+H_1)(3+H_2)n^2k^2+(3n^2k^2+2nk^2)m^{2n}-(n^2k^2-2nk^2)\\
&-n^2K^2(3-H_1)(3-H_2)m^2-(n^2K^2+2nK^2)m^4\}+\omega_0\{-nK(3+H_2)\\
&\times(3n^2K^2-2nK^2)m^{2n+2}-nK(3n^2K^2+2nK^2)(3+H_1)m^{2n}\\
&+nK(n^2K^2-2nK^2)(3-H_1)+nK(n^2K^2+2nK^2)(3-H_2)m^2\}\\
&+\{K^4(9n^4-4n^2)m^{2n}-K^4(n^4-4n^2)\} = 0.
\end{aligned} \tag{8.1}$$

and

$$\omega_1 = F(\omega_0, n, H_1, H_2, m, K) \tag{8.2}$$

We shall find ω_0 from the zeroth order dispersion relation and ω_1 from the first order dispersion relation. We have discussed the following cases :

(*i*) From the zeroth order dispersion relation we find that in general if we express it in the form

$$a\omega_0^4+4b\omega_0^3+6c\omega_0^2+4d\omega_0+e = 0. \tag{8.3}$$

the discriminant of the reducing cubic given by;

$$\Delta = \{I_3 - 27J^2\}. \tag{8.4}$$

where

$$I = \{ae - 4bd + 3c^2\} \tag{8.5}$$

and

$$J = \{ace + 2bcd - ad^2 - eb^2 - c^3\}. \tag{8.6}$$

is less than zero for $m = 1{\cdot}2$, $H_1 = 1{\cdot}2$, $K = 0{\cdot}5$, 5 for $n = 1, 2, 3$ modes. This shows that it has two real and two complex roots. Thus the system is unstable in the zeroth order approximation.

From the first order dispersion relation we note that the effects of low intensity ring currents is to increase the growth rate of instability. Further as K increases the growth rate of instability decreases.

(*ii*) For purely imaginary roots of the zeroth order dispersion relation we set $\omega_0 = i\alpha$ and obtain

$$\alpha = \pm \frac{\begin{array}{r}\{nK^2(3-H_2)(n+2)m^{-n-2} + nK^2(3-H_1)(n-2)m^{-n-4} - nK^2(3n-2) \\ \times(3+H_2)m^{n-2} - nK^2(3+H_1)(3n+2)m^{n-4}\}^{\frac{1}{2}}\end{array}}{\{(3-H_2)m^{-n-2} + (3-H_1)m^{-n} - (3+H_2)m^{n-2} - n(3+H_1)m^{n}\}^{\frac{1}{2}}} \tag{8.7}$$

where

$$\begin{aligned}
&\alpha^4(m^n - m^{-n}) - \alpha^2\{nK^2(3n+2)m^{n-4} + nK^2(3n-2)m^n \\
&+ n^2K^2(3+H_1)(3+H_2)m^{n-2} - nK^2(n-2)m^{-n-4} - nK^2(n+2)m^{-n} \\
&+ n^2K^2(H_1-3)(3-H_2)m^{-n-2}\} + \{n^2K^4(9n^2-4)m^{n-4} \\
&- n^2K^4(n^2-4)m^{-n-4}\} = 0
\end{aligned} \tag{8.8}$$

Thus whenever (8.8) is satisfied, we have one unstable mode. Moreover, ω_1 will increase the growth rate of instability by a fraction of β.

Conclusions

By using the normal mode technique we get more insight of the problem as compared to that of Energy Principle. It was extremely difficult to solve the differential equation satisfied by perturbed quantities in the axisymmetric disturbance for general rotation. However, we got a reasonably good physical picture even when K is of the order of perturbation parameter. The equations involved in θ-disturbance could be solved for general rotation and we find that both the stable and unstable modes were present. Thus, as usual we conclude that the net effect of the θ-disturbance is to bring instability in the device.

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